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CONCERNING ELASTIC CYLINDERS BONDED TO RIGID END PLATES

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Abstract—A right cylinder with plane ends perpendicular to the generators is considered within the context of linear elasticity theory; rigid plates are bonded to the plane ends and one is given a general displacement in its own plane while the other is fixed. Assuming that the material is transversely isotropic with respect to the direction of the generators, it is proved that the strain field converges in energy norm to that of a simple elastic state—the shear state—as a certain parameter, dependent on elastic properties of the cylinder and its thickness—diameter ratio, tends to zero. Various applications are discussed including the case of a thin isotropic cylinder and the case of a cylinder reinforced by fibres aligned parallel to the generators.

INTRODUCTION

Saint-Venant's torsion problem for a right elastic cylinder bounded by plane ends $x_3 = 0$, $x_3 = l$, perpendicular to the generators, may be specified by the boundary conditions

$$u_1 = -\alpha l x_2,\ u_2 = \alpha l x_1,\ \tau_{33} = 0 \quad \text{on } x_3 = l,$$

$$u_1 = u_2 = 0,\ \tau_{33} = 0 \quad \text{on } x_3 = 0,$$

the curved surface being free, and the usual notation being used. The "no end-warping" torsion problem is obtained by replacing the conditions $\tau_{33} = 0$ on $x_3 = 0$, l by $u_3 = 0$ thereon. The torsional rigidity is defined to be the moment of the end tractions about the x_3 axis per unit specific angle of twist α; it is denoted by D in the former case and $\bar{D}$ in the latter. It is known[1, 2], that

$$\mu I \geq \bar{D} \geq D$$

in the case of an isotropic cylinder, where I is the polar moment of inertia about the x_3 axis and μ is the rigidity modulus. It is known[1–3], that

$$\bar{D}/D \to 1$$

as the ratio of the height to cross-sectional length tends to infinity. It was conjectured[2, 3] that

$$\bar{D}/(\mu I) \to 1$$

as the aforementioned ratio tends to zero. It was this conjecture which motivated the present paper. However, a more general proposition is proved and discussed.

A right cylinder, with plane ends perpendicular to the generators, consisting of homogeneous elastic material transversely isotropic with respect to the direction of the generators is considered. Rigid plates are bonded to the plane ends, one of which is given a general displacement in its own plane while the other is kept fixed, and the curved surface is free. The actual elastic state is represented, to begin with, as a linear superposition of two elastic states: the ***shear state*** and the ***difference state***. The ***shear state*** corresponds to the displacement field which varies linearly with the height and coincides with the prescribed displacements; in order to maintain it, tractions have to be applied to the curved surface in general. Suppose that E

denotes Young's modulus for simple tension in the axial direction, μ denotes the shear modulus for shear in planes perpendicular to the generators, while l denotes the height and $\mathscr{L}$ a typical cross-sectional length. The principal result proved is as follows: the actual strain field converges, in *energy norm*, to that of the *shear state* as the dimensionless parameter

$$(\mu/E)(l^2/\mathscr{L}^2)\to 0.$$

Among the consequences of this we have

$$\tilde{D}/(\mu I)\to 1$$

as the aforementioned dimensionless parameter tends to zero.

Clearly there are two important special cases corresponding to the aforementioned dimensionless parameter tending to zero: (1) the isotropic cylinder whose $l/\mathscr{L}$ ratio tends to zero, (2) the transversely isotropic cylinder where $\mu/E\to 0$—corresponding to an idealized cylinder reinforced by continuously (i.e. densely) distributed (almost) inextensible fibres aligned parallel to the generators. These two cases are discussed separately.

The principal references to (1) *supra* in the literature appear to be Synge[4] and Goodier[5]. Synge used a boundary layer type approach to obtain, *inter alia*, an "interior solution" corresponding to the *shear state* in the present instance; Goodier argued the existence of an extended Saint Venant Principle and, by way of illustration, argued that the deviation from the *shear state* is confined to a thin layer in the neighbourhood of the curved boundary. In Section 2 we deduce—from the principal result outlined in the second paragraph—that the strain field of the *difference state* converges to zero in mean square as $l/\mathscr{L}\to 0$: this is equivalent to saying that the actual strain field converges in mean square to that of the *shear state* as $l/\mathscr{L}\to 0$. This is reminiscent of the work of Morgenstern[6] who proved *inter alia* that the strain field associated with the bending of a classical elastic plate is the limit in mean square of the strain field associated with a problem in three-dimensional elasticity as the thickness to cross-sectional length ratio tends to zero. Finally, previous work[3] is used to derive an estimate similar to, yet contrasting with, that described in the last sentence but one as the ratio $l/\mathscr{L}\to 0$.

We now turn to case (2) *supra*. A failry extensive literature concerning idealized fibre-reinforced composites has come into existence over the past decade or so; we mention, in particular, reviews by Spencer[7] and Pipkin[8]. Linear elastic problems involving inextensible fibres, but without incompressibility, have been studied by England, Ferrier and Thomas [9] and Morland[10] in a two-dimensional context. Morland[10] and Everstine and Pipkin[11, 12] have studied certain elastic problems as the limit of inextensibility is approached; the latter article uses a boundary layer approach to a problem which is, to some extent, a two-dimensional analogue of that considered here. In our analysis of case 2 we deduce that the actual strain field converges in mean square to that of the *shear state* as $\mu/E\to 0$. Finally, an expression for the force sustained by the outer or boundary fibres is obtained in particularly simple form.

1. CONVERGENCE IN ENERGY NORM

The usual indicial notation is used, Latin indices running over 1, 2, 3, Greek indices running over 1, 2, and summation being implied by repeated indices; rectangular cartesian coordinates are denoted by x_i.

Consider a right cylinder of homogeneous linear elastic material with plane ends $x_3=0$, $x_3=l$, perpendicular to the generators, whose cross section is $\mathscr{D}$ with boundary $\mathscr{C}$; the material is supposed transversely isotropic with respect to the x_3 direction. Its plane ends are subjected to the displacements

$$u_1=\beta l-\alpha x_2,\quad u_2=\alpha x_1,\quad u_3=0 \qquad \text{on } x_3=l,$$
$$u_1=u_2=u_3=0 \qquad \text{on } x_3=0, \tag{1.1}$$

in the usual notation, while the curved boundary is free and β, α are constants. Such boundary conditions may be simulated in practice by attaching rigid plates to the plane ends $x_3 = l$ and $x_3 = 0$, and by giving the former a general displacement in its own plane while keeping the latter fixed.

The solution to the aforementioned problem is represented by the linear superposition of two elastic states: the *shear state* and the *difference state.*

(a) *Shear state*

Its rectangular cartesian components of displacement, strain and stress are denoted by u_i', e_{ij}', τ_{ij}' respectively. It is characterized by the displacement field

$$u_1' = \beta x_3 - \alpha x_3 x_2, \quad u_2' = \alpha x_3 x_1, \quad u_3' = 0; \tag{1.2}$$

it meets the displacement boundary conditions (1.1) and the traction boundary conditions on the curved boundary $\mathscr{C}$:

$$\tau_{\alpha\beta}'\nu_\beta = 0, \; \tau_{3\beta}'\nu_\beta = -\frac{\mu}{2}\,\mathrm{d}/\mathrm{d}\,s\;\{\alpha(x_1^2 + x_2^2) - 2\beta x_2\} \tag{1.3}$$

where (ν_1, ν_2) are the (x_1, x_2) components of the unit outward normal from $\mathscr{C}$, s denotes arc length measured along $\mathscr{C}$ from some fixed point in the positive direction, and μ denotes the shear modulus corresponding to shear in planes perpendicular to the generators.

(b) *Difference state*

Its rectangular cartesian components of displacement, strain and stress are denoted by u_i'', e_{ij}'', τ_{ij}'' respectively. It corresponds to the boundary conditions

$$u_i'' = 0 \qquad \text{on } x_3 = 0, l$$

and

$$\tau_{\alpha\beta}''\nu_\beta = 0, \; \tau_{3\beta}''\nu_\beta = \frac{\mu}{2}\,\mathrm{d}/\mathrm{d}\,s\;\{\alpha(x_1^2 + x_2^2) - 2\beta x_2\} \qquad \text{on } \mathscr{C}. \tag{1.4}$$

We note that the *shear state* is the actual elastic state if and only if

$$\mathrm{d}/\mathrm{d}\,s\;\{\alpha(x_1^2 + x_2^2) - 2\beta x_2\} = 0$$

(assuming $\mu \neq 0$), or equivalently

$$\mathrm{d}/\mathrm{d}\,s\;\{x_1^2 + (x_2 - \beta/\alpha)^2\} = 0, \qquad (\alpha \neq 0). \tag{1.5}$$

That is, if and only if the boundary consists of a circle whose centre is $(0, \beta/\alpha)$, or two concentric circles with the aforementioned point as centre: this corresponds to the well known result that torsion without warping can only occur for cross sections which are circular, or for those bounded by concentric circles (e.g. [13]).

It is not difficult to establish, using the Reciprocal Theorem, that

$$\mathscr{E} = \mathscr{E}' - \mathscr{E}'' \tag{1.6}$$

where $\mathscr{E}$ denotes the strain energy corresponding to the actual elastic state, $\mathscr{E}'$ that corresponding to the *shear state*, and $\mathscr{E}''$ that corresponding to the *difference state.*

We now obtain an upper bound for $\mathscr{E}''$. It proves convenient to write

$$f(s) = \frac{1}{2}\,\mathrm{d}/\mathrm{d}\,s\;\{\alpha(x_1^2 + x_2^2) - 2\beta x_2\} \tag{1.7}$$

defined on the boundary $\mathscr{C}$. In the present context, the Principle of Minimum Complementary Energy gives

$$\mathscr{E}'' \leq \int_V W(\sigma_{ij})\, \mathrm{d}\, V, \tag{1.8}$$

where σ_{ij} is any statically admissible stress field: one which is symmetric, and satisfies

$$\left.\begin{array}{ll}\text{(a) } \sigma_{ij,j} = 0 & \text{in } V \text{ (domain of cylinder)},\\ \text{(b) } \sigma_{\alpha\beta}\nu_\beta = 0,\ \sigma_{3\beta}\nu_\beta = \mu f(s) & \text{on } \mathscr{C},\\ \text{(c) } \sigma_{ij} \epsilon C^1(\bar{V}), & \end{array}\right\} \tag{1.9}$$

and where $W(\sigma_{ij})$ denotes the strain energy density corresponding to the stress field σ_{ij}. A stress field of the type outlined is generated by the function

$$\Theta(x_1, x_2) \epsilon C^3(\mathscr{D} \cup \mathscr{C})$$

where†

$$\sigma_{\alpha 3} = \mu\Theta_{,\alpha}, \quad \sigma_{33} = -\mu(x_3 - l/2)\nabla_1{}^2\Theta,$$
$$\sigma_{ij} = 0 \quad \text{(otherwise)}, \tag{1.10}$$

and

$$\partial\Theta/\partial\nu = f(s) \quad \text{on } \mathscr{C}, \tag{1.11}$$

the latter derivative signifying differentiation along the outward normal from $\mathscr{C}$. An elementary calculation shows that

$$\int_V W(\sigma_{ij})\, \mathrm{d} V = \frac{1}{2}\mu l \int_{\mathscr{D}} \{(\nabla_1\Theta)^2 + (\lambda\mathscr{L})^2(\nabla_1{}^2\Theta)^2\}\, \mathrm{d}A \tag{1.12}$$

where

$$\lambda = \{(\mu/E)(l^2/\mathscr{L}^2)/12\}^{1/2} \tag{1.13}$$

$\mathscr{L}$ being a typical cross-sectional length—the perimeter length, say—and E being Young's modulus corresponding to simple tension in the axial direction. The best bound (1.8) of the type envisaged is furnished by that function Θ which satisfies

$$(\lambda\mathscr{L})^2\nabla_1{}^4\Theta - \nabla_1{}^2\Theta = 0 \quad \text{in } \mathscr{D} \tag{1.14}$$

—the Euler–Lagrange differential equation corresponding to (1.12)—subject to the natural boundary condition

$$(\lambda\mathscr{L})^2\partial(\nabla_1{}^2\Theta)/\partial\nu - \partial\Theta/\partial\nu = 0 \quad \text{on } \mathscr{C} \tag{1.15}$$

together with (1.11). The solution of the one-dimensional analogue of this two-dimensional system gives the clue to the choice of Θ that follows.

The coordinate system (s, n) is adopted, where s is, as usual, arc length measured along $\mathscr{C}$ from some fixed point in the positive direction, and n is the distance measured along an inwardly drawn normal to $\mathscr{C}$; such a coordinate system is unique provided n is less than a

†The particular linear dependence of σ_{33} on x_3 is suggested by symmetry.

certain length characteristic of the cross section (e.g. [14]). Let

$$\Theta = \lambda\mathscr{L}(1+4\lambda^{1-\delta})^{-1}f(s)\,\bar{e}^{n/\lambda\mathscr{L}}\{1-n/(\lambda^{\delta}\mathscr{L})\}^4 \text{ for } n<\lambda^{\delta}\mathscr{L},$$
$$\Theta = 0 \quad \text{otherwise,} \tag{1.16}$$

where λ is sufficiently small and δ is some number such that $1>\delta>0$. If we assume that the parametric equations of the boundary $\mathscr{C}: x_\alpha = x_\alpha(s)$ are four times continuously differentiable, then Θ satisfies all the conditions stipulated in the last paragraph. Now

$$\int_{\mathscr{D}}\{(\nabla_1\Theta)^2+(\lambda\mathscr{L})^2(\nabla_1{}^2\Theta)^2\}\,\mathrm{d}A \sim \int_{L_\lambda}\{(\partial\Theta/\partial n)^2+(\lambda\mathscr{L})^2(\partial^2\Theta/\partial n^2)^2\}\,\mathrm{d}A \qquad \text{as } \lambda\to 0 \tag{1.17}$$

where L_λ denotes the layer wherein Θ is non-zero. Further,

$$\int_{L_\lambda}(\partial\Theta/\partial n)^2\,\mathrm{d}A \sim \int_{\mathscr{C}} f^2(s)\,\mathrm{d}s\int_0^{\lambda^{\delta}\mathscr{L}}\bar{e}^{2n/(\lambda\mathscr{L})}\,\mathrm{d}n$$
$$= (\lambda\mathscr{L}/2)\int_{\mathscr{C}} f^2(s)\,\mathrm{d}s\,(1-\bar{e}^{2/\lambda^{1-\delta}})$$
$$\sim (\lambda\mathscr{L}/2)\int_{\mathscr{C}} f^2(s)\,\mathrm{d}s \qquad \text{as } \lambda\to 0. \tag{1.18}$$

Similarly

$$\int_{L_\lambda}(\lambda\mathscr{L})^2(\partial\Theta/\partial n^2)^2\,\mathrm{d}A \sim (\lambda\mathscr{L}/2)\int_{\mathscr{C}} f^2(s)\,\mathrm{d}s \qquad \text{as } \lambda\to 0. \tag{1.19}$$

Consequently

$$\int_{\mathscr{D}}\{(\nabla_1\Theta)^2+(\lambda\mathscr{L})^2(\nabla_1{}^2\Theta)^2\}\,\mathrm{d}A \sim \lambda\mathscr{L}\int_{\mathscr{C}} f^2(s)\,\mathrm{d}s \qquad \text{as } \lambda\to 0. \tag{1.20}$$

In view of (1.8), (1.12) and (1.20) we obtain

$$\mathscr{E}'' \le (\mu l/2)\int_{\mathscr{C}} f^2(s)\,\mathrm{d}s\cdot\lambda\mathscr{L}\{1+g(\lambda)\} \qquad \text{as } \lambda\to 0 \tag{1.21}$$

where $g(\lambda)$ is a function of λ such that $g(\lambda)\to 0$ as $\lambda\to 0$. It is readily verified that

$$\mathscr{E}''/\mathscr{E}' \le \{(\mathscr{L}/4)\int_{\mathscr{C}} R^2(\mathrm{d}R/\mathrm{d}s)^2\,\mathrm{d}s/\int_{\mathscr{D}} R^2\,\mathrm{d}A\}\cdot\lambda\{1+g(\lambda)\} \qquad \text{as } \lambda\to 0 \tag{1.22}$$

where $R=\{x_1{}^2+(x_2-\beta/\alpha)^2\}^{1/2}$ – the distance of a current point from the centre of rotation. We note that the "geometrical factor" in (1.22) remains finite even when the centre of rotation. recedes to infinity (pure translation). However, this quantity becomes very large for certain strip-like cross sections whose breadth-length ratio is very small; such limiting cross sections are excluded from subsequent considerations. The result (1.22) may be stated in the form

$$\mathscr{E}''/\mathscr{E}' = O(\lambda) \qquad \text{as } \lambda\to 0 \tag{1.23}$$

or, equivalently,

$$\mathscr{E}''/\mathscr{E} = O(\lambda) \qquad \text{as } \lambda\to 0 \tag{1.24}$$

on using (1.6). The foregoing may be expressed in substance as follows: "the actual strain field converges in *energy norm* to the strain field of the *shear state* as the dimensionless parameter λ tends to zero".

It follows from (1.23) and (1.6) that

$$\mathscr{E}/\mathscr{E}' \to 1 \qquad \text{as } \lambda \to 0. \tag{1.25}$$

Writing $\beta = 0$ in (1.1) we have

$$\mathscr{E} = \bar{D}\alpha^2 l/2, \quad \mathscr{E}' = \mu I \alpha^2 l/2, \tag{1.26}$$

$\bar{D}$ being the torsional rigidity (no end-warping) and I being the polar moment of inertia of the cross section about the x_3 axis; it follows from (1.25) and (1.26) that

$$\bar{D}/(\mu I) \to 1 \qquad \text{as } \lambda \to 0. \tag{1.27}$$

This result was conjectured for an isotropic cylinder $l/\mathscr{L} \to 0$ in[2, 3]. We note that (1.21) provides a lower bound for $\bar{D}$ when λ is small, on noting (1.6) and (1.26).

2. SHORT ISOTROPIC CYLINDER; FIBRE-REINFORCED CYLINDER

To examine some further implications of (1.22)–(1.24) it proves convenient to examine separately (1) the case of a thin isotropic cylinder $l/\mathscr{L} \ll 1$, (2) the case of a transversely isotropic cylinder for which (a) $\mu/E \ll 1$, (b) all stiffnesses other than E are comparable with μ, and (c) l and $\mathscr{L}$ are comparable. Case 2 corresponds to a model of a cylinder reinforced by almost inextensible fibres continuously (densely) distributed therein and aligned parallel to the generators.

Case 1.

Introducing the "reference" strain e′ associated with the *shear state* by

$$\mathscr{E}' = \mu\, e'^2 V \tag{2.1}$$

where V denotes the volume of the cylinder, we define the "scaled" strains of the *difference state* by

$$E''_{ij} = e''_{ij}/e'. \tag{2.2}$$

Assuming that the Lamé constants λ, μ are both positive, the strain-energy density $W(e_{ij})$ satisfies

$$2W(e_{ij}) \geq \mu e_{ij} e_{ij}. \tag{2.3}$$

It follows from (1.23) and (2.1) to (2.3) that

$$\int_V E''_{ij} E''_{ij}\, \mathrm{d}V/V = O(l/\mathscr{L}) \qquad \text{as } l/\mathscr{L} \to 0. \tag{2.4}$$

The substance of this may be expressed: "the actual strain field converges in mean square to the strain field associated with the *shear state* as the ratio of the height to cross-sectional length tends to zero".

This result indicates that, for $l/\mathscr{L} \ll 1$, the actual strain field differs but slightly from that of the *shear state* throughout—apart from a small region/small regions where the difference may be significant: such a region occurs in the neighbourhood of the curved surface since the *shear state* is incompatible with the boundary conditions thereon in general. We note that an extended Saint-Venant principle, analogous to that established in[15], is applicable to the *difference state*: it indicates rapid decay of the elastic field as one comes in from the boundary; see also [5].

We use some previous work [3] to contrast the foregoing with what happens when $l/\mathscr{L} \to \infty$. Write $\beta = 0$ in connection with $(1.1)_1$, and introduce the *Saint-Venant state* whose displacement components are denoted by $\check{u}_i$: it is defined by the boundary conditions

$$\check{u}_1 = -\alpha l x_2, \quad \check{u}_2 = \alpha l x_1, \quad \check{u}_3 = \phi(x_1, x_2) \qquad \text{on } x_3 = l,$$
$$\check{u}_1 = \check{u}_2 = 0, \quad \check{u}_3 = \phi(x_1, x_2) \qquad \text{on } x_3 = 0, \tag{2.5}$$

together with zero traction on the curved boundary; $\phi(x_1, x_2)$ is the Saint-Venant warping function satisfying

$$\nabla_1^2\phi = 0 \quad \text{in } \mathscr{D},\ \partial\phi/\partial\nu = \mathrm{d}/\mathrm{d}s\{(x_1^2 + x_2^2)/2\} \text{on } \mathscr{C}. \tag{2.6}$$

The actual elastic state is now represented as the superposition of the *Saint-Venant state* and the (*second*) *difference state*. Denoting the strain-energy associated with the *Saint-Venant state* by $\check{\mathscr{E}}$ and that associated with the (*second*) *difference state* by $\bar{\mathscr{E}}$, the result

$$\mathscr{E} = \check{\mathscr{E}} + \bar{\mathscr{E}} \tag{2.7}$$

holds; using (2.7) and the results of [3] it may be established that

$$\bar{\mathscr{E}} \leq \sqrt{7}(\mu I - \mathrm{D})\alpha^2\Lambda^{-1} \tag{2.8}$$

where D denotes the *Saint-Venant* torsional rigidity and Λ the lowest non-zero frequency of a free membrane coinciding with the cross section. Denoting the strain components of the (*second*) *difference state* by $\bar{e}_{ij}$, and defining a "reference" strain $\check{e}$ for the *Saint-Venant state by*

$$\mu\check{e}^2 V = \check{\mathscr{E}} = D\alpha^2 l/2 \tag{2.9}$$

we define the "scaled" strain components

$$\bar{E}_{ij} = \bar{e}_{ij}/\check{e}. \tag{2.10}$$

It follows from (2.8) and (2.9) that

$$\bar{\mathscr{E}}/(\mu\check{e}^2 V) = 2\sqrt{7}\{(\mu I/D) - 1\}/(\Lambda l). \tag{2.11}$$

Confining attention to the case where the distance of the centre of rotation from the centroid of the cross section is of the order $\mathscr{L}$ at most, and excluding thin cross sections with very small breadth/length ratio, we find, using (2.3) and (2.11) that

$$\int_V \bar{E}_{ij}\bar{E}_{ij}\,\mathrm{d}V_V = \int_V \bar{E}_{ij}\bar{E}_{ij}\,\mathrm{d}V/V = 0(\mathscr{L}/l) \qquad \text{as } l/\mathscr{L} \to \infty \tag{2.12}$$

(see (2.4)). The substance of this may be expressed "the actual strain field converges in mean square to that of the *Saint-Venant state* as the ratio of the height to cross-sectional length tends to infinity". This is reminiscent of Saint-Venant's principle.

Case 2.

The result

$$\int_V E''_{ij}E''_{ij}\,\mathrm{d}V/V = 0\{(\mu/E)^{1/2}\} \qquad \text{as } \mu/E \to 0 \tag{2.13}$$

analogous to (2.4), is easily established.

If the stress-strain relations for the transversely isotropic material are given by

$$\begin{aligned} \tau_{11} &= C_{11}e_{11} + C_{12}e_{22} + C_{13}e_{33}, & \tau_{12} &= (C_{11} - C_{12})e_{12}, \\ \tau_{22} &= C_{12}e_{11} + C_{11}e_{22} + C_{13}e_{33}, & \tau_{23} &= 2\mu e_{23}, \\ \tau_{33} &= C_{13}(e_{11} + e_{22}) + C_{33}e_{33}, & \tau_{13} &= 2\mu e_{13}, \end{aligned} \tag{2.14}$$

then the left-hand side of (1.23) can be expressed in the form

$$\begin{aligned} \mathscr{E}''/\mathscr{E}' = \frac{1}{2}\int_V \Big[\frac{1}{2}(C_{11} - C_{12})(E''_{11} - E''_{22})^2 + \Big\{\frac{1}{2}(C_{11} + C_{12}) \\ - C_{13}^2/C_{33}\Big\}(E''_{11} + E''_{22})^2 + C_{33}\{E''_{33} + (C_{13}/C_{33})(E''_{11} + E''_{22})\}^2 \\ + 2(C_{11} - C_{12})E''^2_{12} + 4\mu(E''^2_{13} + E''^2_{23})\Big]\,\mathrm{d}V/(\mu V). \end{aligned} \tag{2.15}$$

The positive-definiteness of the strain-energy density implies the following inequalities *inter alia*

$$C_{11} - C_{12} > 0,\ C_{11} + C_{12} > 0,\ C_{33} > 0,\ \mu > 0. \tag{2.16}$$

Using (1.23), (2.15), (2.16), and remembering that all the stiffnesses (moduli) in (2.14) are comparable with μ—apart from C_{33} which is comparable with E—we obtain the following mean square estimates

$$\begin{aligned} &\|E''_{ij}\|^2 = 0\{(\mu/E)^{1/2}\} \qquad (i \neq j), \\ &\|E''_{11} - E''_{22}\|^2 = 0\{(\mu/E)^{1/2}\}, \\ &\|E''_{11} + E''_{22}\|^2 = 0\{(\mu/E)^{1/2}\}, \\ &\|E''_{33} + (C_{13}/C_{33})(E''_{11} + E''_{22})\|^2 = 0\{(\mu/E)^{3/2}\} \qquad \text{as } \mu/E \to 0; \end{aligned} \tag{2.17}$$

the notation

$$\|f\| = \left\{\int_V f^2\,\mathrm{d}V/V\right\}^{1/2} \tag{2.18}$$

is used in the foregoing and subsequently. The second and third of the estimates (2.17) together imply

$$\|E''_{11}\|^2 = 0\{(\mu/E)^{1/2}\},\ \|E''_{22}\|^2 = 0\{(\mu/E)^{1/2}\} \qquad \text{as } \mu/E \to 0. \tag{2.19}$$

Now

$$\begin{aligned} \|E''_{33}\| &\leq \|E''_{33} + (C_{13}/C_{33})(E''_{11} + E''_{22})\| + \|-(C_{13}/C_{33})(E''_{11} + E''_{22})\| \\ &\leq \|E''_{33} + (C_{13}/C_{33})(E''_{11} + E''_{22})\| + |C_{13}/C_{33}|(\|E''_{11}\| + \|E''_{22}\|). \end{aligned} \tag{2.20}$$

Using $(2.17)_4$, (2.19) and (2.20), and remembering that C_{13} is comparable with μ, in common with all other stiffnesses (moduli) in (2.14) other than C_{33} which is comparable with E, we obtain the estimate

$$\|E''_{33}\|^2 = 0\{(\mu/E)^{3/2}\} \qquad \text{as } \mu/E \to 0; \tag{2.21}$$

the preceding work shows that, for all other values of the indices i, j, we have

$$\|E''_{ij}\|^2 = 0\{(\mu/E)^{1/2}\} \qquad \text{as } \mu/E \to 0. \tag{2.22}$$

The substance of the foregoing estimates may be expressed: "the actual strain field converges in mean square to that of the *shear state* as the ratio of the relevant rigidity modulus to Young's modulus tends to zero".

The foregoing estimates indicate that the actual strain field coincides with that of the *shear state* in the limit $\mu/E \to 0$ except possibly in a set of measure zero: such a set occurs on (or adjacent to) the curved boundary as the *shear state* is incompatible with the boundary conditions thereon in general. Making the plausible assumption that the aforementioned is the only such set, it follows the stresses of the *difference state* (suitably non-dimensionalized with respect to μ) τ''_{ij}/μ are all zero in the open cylinder except, possibly, τ''_{33}/μ. The latter can be shown to be zero throughout the open cylinder using (a) the fact that τ''_{13}/μ, τ''_{23}/μ are zero in the limit, (b) equilibrium considerations, and (c) the fact that τ''_{33} is zero on the plane $x_3 = l/2$ from symmetry considerations.

We conclude by finding the force sustained by the outer or boundary fibres—the most interesting quantity, perhaps, associated with this limiting problem.

For the moment we confine attention to the limiting case ($\mu/E \to 0$) of the *difference state*. In order that a non-zero shear stress

$$\tau''_{\alpha 3}\nu_\alpha = \mu f(s) \tag{2.23}$$

on the curved boundary may coexist with zero stress field within, there must necessarily be a singularity in τ''_{33}. We can deduce an expression for this or, rather, for the force, per unit length of boundary $\mathscr{C}$, which must be sustained by the fibres on the curved boundary; this is done by considering the equilibrium of an infinitesimal element of the cylinder bounded by the curved boundary $\mathscr{C}$ and by a parallel boundary a distance δn from it, by two planes normal to the curved boundary distant δs from one another, and such that it lies within $(x_3, x_3 + \delta x_3)$. If δF denotes the force, sustained by the fibres per unit length of $\mathscr{C}$, we have

$$\delta F \delta s + \delta x_3 \delta s \mu f(s) = 0 \tag{2.24}$$

or, in the limit,

$$\mathrm{d}F/\mathrm{d}x_3 = -\mu f(s), \tag{2.25}$$

whence

$$F = -\mu(x_3 - l/2) f(s), \tag{2.26}$$

on noting that $F = 0$ at $x_3 = l/2$ from symmetry considerations. In the foregoing, positive values of F correspond to tension.

Recalling that the actual elastic state is the superposition of the *shear state* and the *difference state*, it follows that (2.26) gives the actual force, per unit length of the curved boundary sustained by the boundary fibres. Consider now the total force $F_{1,2}$ sustained by the fibres on the portion of the boundary between the points P_1, P_2 as one travels from P_1 to P_2 in the positive sense. If $\overset{2}{r}$ and $\overset{1}{r}$ denote the distances of P_2 and P_1 respectively from the x_3 axis and if $\overset{2}{x}_2$, $\overset{1}{x}_2$ denote the respective values of the coordinate x_2, and if we integrate (2.26) and recall (1.7), we obtain the simple expression

$$F_{1,2} = -\frac{1}{2}\mu(x_3 - l/2)\{\alpha(\overset{2}{r}{}^2 - \overset{1}{r}{}^2) - 2\beta(\overset{2}{x}_2 - \overset{1}{x}_2)\}. \tag{2.27}$$

We note that the result (2.26) is obtainable by integrating the "approximate" stress σ_{33} across the layer L_λ and letting $\lambda \to 0$.

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